ON THE LIE ALGEBRAS ASSOCIATED WITH PURE MAPPING CLASS GROUPS

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ABSTRACT. Pure braid groups and pure mapping class groups of a punctured sphere have many features in common. In the paper the graded Lie algebra of the descending central series of the pure mapping class of a sphere is studied. A simple presentation of this Lie algebra is obtained.

CONTENTS

1. INTRODUCTION

Mapping class group is an important object in Topology, Complex Analysis, Algebraic Geometry and other domains. It is a lucky case when the method of Algebraic Topology works perfectly well, the application of the functor of fundamental group completely solves the topological problem: group of isotopy classes of homeomorphisms is described in terms of automorphisms of the fundamental group of the corresponding surface, as states the Dehn-Nilsen-Baer theorem, see [15], for example.

Let $S_{g,b,n}$ be an oriented surface of the genus g with b boundary components and with a set Q_n of n fixed points. Consider the group $\mathrm{Homeo}(S_{g,b,n})$ of orientation preserving self-homeomorphisms of $S_{g,b,n}$ which fix pointwise the boundary (if it exists) and map the set Q_n into itself. Orientation reversing homeomorphisms also possible to consider, see [8], for example, but we restrict ourselves to orientation preserving case. Let $\mathrm{Homeo}^0(S_{g,b,n})$ be the normal subgroup of self-homeomorphisms of $S_{g,b,n}$ which are isotopic to identity. Then the *mapping class group* $\mathcal{M}_{g,b,n}$ is defined as a factor group

$$\mathcal{M}_{g,b,n} = \mathrm{Homeo}(S_{g,b,n})/\,\mathrm{Homeo}^0(S_{g,b,n}).$$

These groups are connected closely with braid groups. In [19] W. Magnus interpreted the braid group as the mapping class group of a punctured disc with the fixed boundary:

$$Br_n \cong \mathcal{M}_{0,1,n}.$$

The same way as the braid groups the group $\mathcal{M}_{g,b,n}$ has a natural epimorphism to the symmetric group Σ_n with the kernel called the *pure mapping class group* $\mathcal{PM}_{g,b,n}$, so there exists an exact

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sequence:

$$1 \to \mathcal{PM}_{g,b,n} \to \mathcal{M}_{g,b,n} \to \Sigma_n \to 1.$$

Geometrically pure mapping class group $\mathcal{PM}_{g,b,n}$ is descried as consisting of isotopy classes of homeomorphisms that preserve the punctures pointwise.

We call an element m of the pure mapping class group $\mathcal{PM}_{g,b,n}$ *i-Makanin* or *i-Brunnian* if a homeomorphism h lying in the class m

$$h : S_{g,b,n} \to S_{g,b,n},$$

becomes isotopical to the identity map if we fill the deleted point i (or if it becomes non-fixed). Filling the point i generates the homomorphism

$$pm_i : \mathcal{PM}_{g,b,n} \to \mathcal{PM}_{g,b,n-1}.$$

We denote the subgroup of i-Makanin elements of the mapping class group by A_i, it is so the kernel of pm_i. The subgroups A_i, $i = 1, \dots, n$, are conjugate in $\mathcal{M}_{g,b,n}$. The intersection of all subgroups of i-Makanin elements is the of *Makanin* or *Brunnian* subgroup of the mapping class group

$$Mak_{g,b,n} = \cap_{i=1}^{n} A_i.$$

In the paper we consider the pure mapping class group of a sphere with no boundary components $\mathcal{PM}_{0,0,n}$, which we denote for simplicity by $\mathcal{PM}_{0,n}$. We study the natural Lie algebra obtained from the descending central series for $\mathcal{PM}_{0,n}$. One motivation for the work here is that the group $\mathcal{PM}_{0,n}$ is natural as well as accessible case continuing the same study for the pure braid group done in the works of T. Kohno [17], T. Kohno and T. Oda [18], Y. Ihara [16], R. Bezrukavnikov [4] and for McCool group it was done in the works [7] and [2].

2. Lie algebra $gr^*(\mathcal{PM}_{0,n})$

The group $\mathcal{PM}_{0,n}$ is closely related to the pure braid group on a sphere $P_n(S^2)$ as well as its non-pure analogue $\mathcal{M}_{0,n}$ is connected with the (total) braid group of a sphere $Br_n(S^2)$.

We start with presentations. Usually the braid group Br_n is given by the following Artin presentation [1]. It has the generators σ_i, $i = 1, ..., n-1$, and two types of relations:

$$\begin{cases} \sigma_i \sigma_j &= \sigma_j \sigma_i, \quad \text{if} \quad |i-j| > 1, \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}\,. \end{cases} \tag{2.1}$$

The generators $a_{i,j}$, $1 \le i < j \le n$ for the pure braid group P_n (of a disc) can be defined (as elements of the the braid group Br_n) by the formula:

$$a_{i,j} = \sigma_{j-1} ... \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} ... \sigma_{j-1}^{-1}.$$

Then the defying relations, which are called the *Burau relations* [5], [22] are as follows:

$$\begin{cases} a_{i,j} a_{k,l} = a_{k,l} a_{i,j} \text{ for } i < j < k < l \text{ and } i < k < l < j, \\ a_{i,j} a_{i,k} a_{j,k} = a_{i,k} a_{j,k} a_{i,j} \text{ for } i < j < k, \\ a_{i,k} a_{j,k} a_{i,j} = a_{j,k} a_{i,j} a_{i,k} \text{ for } i < j < k, \\ a_{i,k} a_{j,k} a_{j,l} a_{j,k}^{-1} = a_{j,k} a_{j,l} a_{j,k}^{-1} a_{i,k} \text{ for } i < j < k < l. \end{cases} \tag{2.2}$$

It was proved by O. Zariski [26] and then rediscovered by E. Fadell and J. Van Buskirk [9] that a presentation for the braid group of a sphere can be given with the generators σ_i, $i = 1, ..., n-1$,

the same as for the classical braid group, satisfying the braid relations (2.1) and the following sphere relation:

$$\sigma_1\sigma_2\dots\sigma_{n-2}\sigma_{n-1}^2\sigma_{n-2}\dots\sigma_2\sigma_1 = 1. \tag{2.3}$$

Having the same generators, but if we add to the braid relations (2.1) and the sphere relation (2.3) one more relation ((2.4) below) we get the presentation for the mapping class group of a punctured sphere $\mathcal{M}_{0,n}$ obtained by W. Magnus [19], see also [20] and [21].

$$(\sigma_1\sigma_2\dots\sigma_{n-2}\sigma_{n-1})^n = 1. \tag{2.4}$$

Let Δ be the Garside's fundamental word in the braid group Br_n [12]. It can be in particular expressed by the formula:

$$\Delta = \sigma_1\dots\sigma_{n-1}\sigma_1\dots\sigma_{n-2}\dots\sigma_1\sigma_2\sigma_1.$$

If we use Garside's notation $\Pi_t \equiv \sigma_1\dots\sigma_t$, then $\Delta \equiv \Pi_{n-1}\dots\Pi_1$. If the generators σ_1, σ_2, $\dots$, σ_{n-2}, σ_{n-1}, are subject to the braid relations (2.1), then the condition (2.4) is equivalent to the following relation

$$\Delta^2 = 1.$$

For the pure braid group on a sphere let us introduce the elements $a_{i,j}$ for all i, j by the formulas:

$$\begin{cases} a_{j,i} = a_{i,j} & \text{for } i < j \le n, \\ a_{i,i} = 1. \end{cases} \tag{2.5}$$

The pure braid group for the sphere has the generators $a_{i,j}$ which satisfy Burau relations (2.2), relations (2.5), and the following relations [13]:

$$a_{i,i+1}a_{i,i+2}\dots a_{i,i+n-1} = 1 \quad \text{for all } i \le n,$$

with the convention that $k + n = k$. Note that Δ^2 is a pure braid and it can be expressed by the following formula

$$\Delta^2 = (a_{1,2}a_{1,3}\dots a_{1,n})(a_{2,3}a_{2,4}\dots a_{2,n})\dots(a_{n-1,n}) = \\ (a_{1,2})(a_{1,3}a_{2,3})(a_{1,4}a_{2,4}a_{3,4})\dots(a_{1,n}\dots\dots a_{n-1,n}).$$

The fact that this element of the braid group generates its center goes back to Chow [6].

Let us denote by $P_n(S^2_3)$ the pure braid group on n strings of a sphere with three points deleted or equivalently the subgroup of the pure braid group of a disc on $n+2$ strings where (say, the last) two strings are fixed.

The following statement follows from the normal forms of the groups $P_n(S^2)$ and $\mathcal{PM}_{0,n}$ [13] and on the geometrical level it was expressed in [14]. Note that the groups $P_2(S^2)$ and $\mathcal{PM}_{0,3}$ are trivial.

Theorem 2.1. *(i) The pure braid group of a sphere $P_n(S^2)$ for $n \ge 3$ is isomorphic to the direct product of the cyclic group of order 2 (generated by Δ^2) and $\mathcal{PM}_{0,n}$.*

(ii) The pure braid group P_n for $n \ge 2$ is isomorphic to the direct product of the infinite cyclic group (generated by Δ^2) and $\mathcal{PM}_{0,n+1}$.

(iii) The groups $\mathcal{PM}_{0,n}$ and $P_{n-3}(S^2_3)$ are isomorphic for $n \ge 4$.

The isomorphism of the part (i) of Theorem 2.1 is compatible with the homomorphisms $p_i : P_n(S^2) \to P_{n-1}(S^2)$, $pm_i : \mathcal{PM}_{0,n} \to \mathcal{PM}_{0,n-1}$ consisting of deleting one string or forgetting one point, so the group of Makanin braids of a sphere coincide with the subgroup of Makanin mapping class of a sphere.

For a group G the descending central series

$$G = \Gamma_1 > \Gamma_2 > \cdots > \Gamma_i > \Gamma_{i+1} > \ldots .$$

is defined by the formulas

$$\Gamma_1 = G, \quad \Gamma_{i+1} = [\Gamma_i, G].$$

The descending central series of a discrete group G gives rise to the associated graded Lie algebra (over $\mathbb{Z}$) $gr^*(G)$ [24].

$$gr^i(G) = \Gamma_i/\Gamma_{i+1}.$$

A presentation of the Lie algebra $gr^*(P_n)$ for the pure braid group can be described as follows [17]. It is the quotient of the free Lie algebra $L[A_{i,j}|\, 1 \leq i < j \leq n]$ generated by elements $A_{i,j}$ with $1 \leq i < j \leq n$ modulo the "infinitesimal braid relations" or "horizontal $4T$ relations" given by the following three relations:

$$\begin{cases} [A_{i,j}, A_{s,t}] = 0, \text{ if } \{i,j\} \cap \{s,t\} = \phi, \\ [A_{i,j}, A_{i,k} + A_{j,k}] = 0, \text{ if } i < j < k, \\ [A_{i,k}, A_{i,j} + A_{j,k}] = 0, \text{ if } i < j < k. \end{cases}$$

Y. Ihara in [16] gave a presentation of the Lie algebra $gr^*(P_n(S^2))$ of the pure braid group of a sphere. It is convenient to have conventions like (2.5). So, it is the quotient of the free Lie algebra $L[B_{i,j}|\, 1 \leq i,j \leq n]$ generated by elements $B_{i,j}$ with $1 \leq i,j \leq n$ modulo the following relations:

$$\begin{cases} B_{i,j} = B_{j,i} \text{ for } 1 \leq i,j \leq n, \\ B_{i,i} = 0 \ \text{ for } 1 \leq i \leq n, \\ [B_{i,j}, B_{s,t}] = 0, \text{ if } \{i,j\} \cap \{s,t\} = \phi, \\ \sum_{j=1}^{n} B_{i,j} = 0, \text{ for } 1 \leq i \leq n. \end{cases}$$

It is a factor algebra of the algebra $gr^*(P_n)$: the last two relations in (2) are the consequences of the third and the forth type relations in (2).

Theorem 2.2. *(i) The graded Lie algebra $gr^*(\mathcal{PM}_{0,n})$ is the quotient of the free Lie algebra $L[B_{i,j}|\, 1 \leq i,j \leq n]$ modulo the following relations:*

$$\begin{cases} B_{i,j} = B_{j,i} \textit{ for } 1 \leq i,j \leq n, \\ B_{i,i} = 0 \ \textit{ for } 1 \leq i \leq n, \\ [B_{i,j}, B_{s,t}] = 0, \ \textit{if } \{i,j\} \cap \{s,t\} = \phi, \\ \sum_{j=1}^{n} B_{i,j} = 0, \ \textit{for } 1 \leq i \leq n, \\ \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} B_{i,j} = 0. \end{cases} \tag{2.6}$$

(ii) The graded Lie algebra $gr^*(\mathcal{PM}_{0,n})$ is the quotient of the free Lie algebra $L[A_{i,j}|\, 1 \leq i < j \leq n-1]$ generated by elements $A_{i,j}$ with $1 \leq i < j \leq n-1$ modulo the following relations:

$$\begin{cases} [A_{i,j}, A_{s,t}] = 0, \ \textit{if } \{i,j\} \cap \{s,t\} = \phi, \\ \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} A_{i,j} = 0. \end{cases}$$

Proof. Part (i) of Theorem 2.6 follows from the Ihara presentation and the part (i) of Theorem 2.1. The pure mapping class group $\mathcal{PM}_{0,n}$ is a direct summand in the pure braid group of a sphere $P_n(S^2)$, so there is no problem in obtaining exact sequence after application of the functor of associated graded Lie algebras gr^*.

To obtain part (ii) let us write in detail the system of linear equations which constitute the forth part of the Ihara relations (2):

$$\begin{cases} B_{1,2} + B_{1,3} + \dots B_{1,n} = 0, \\ B_{1,2} + B_{2,3} + \dots B_{2,n} = 0, \\ \dots \\ B_{1,n} + B_{2,n} + \dots B_{n-1,n} = 0. \end{cases} \tag{2.7}$$

The $n-1$ equations (except the last one) give the possibility to exclude the letters $B_{i,n}$ for $1 \leq i \leq n-1$ from the presentation. Note that the third type relations in (2) with $j = n$ are the consequences of the same type relation with $j \leq n-1$ and relations (2.7) except the last one. Take then the linear combination of these equations where the first $n-1$ equations are taken with the coefficient $+1$ and the last one with the coefficient -1. We get the equation

$$2(\sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} A_{i,j}) = 0.$$

The second type relation in (2.6) is a consequence of the last equation in (2) and first relation in (2.7). □

Corollary 2.1. *A presentation of the Lie algebra $gr^*(P_n(S^2))$ can be given with generators $A_{i,j}$ with $1 \leq i < j \leq n-1$, modulo the following relations:*

$$\begin{cases} [A_{i,j}, A_{s,t}] = 0, \ \ if \ \ \{i,j\} \cap \{s,t\} = \phi, \\ 2(\sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} A_{i,j}) = 0. \end{cases}$$

So, the element $\sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} A_{i,j}$ of order 2 generates the central subalgebra in $gr^*(P_n(S^2))$.

□

3. Example

The pure braid group of a sphere $P_4(S^2)$ is isomorphic to the direct product of the cyclic group of order 2 (generated by Δ^2) and the pure braid group on one string of a sphere with three points deleted, that is the fundamental group of disc with two points deleted, that is a free group on two generators F_2. Its associated graded Lie algebra is a direct sum of central $\mathbb{Z}/2$ and the free Lie algebra on two generators. The pure mapping class group $\mathcal{PM}_{0,4}$ is isomorphic to a free group on two generators. According to Theorem 2.2 its associated graded Lie algebra is the quotient of the free Lie algebra $L[A_{1,2}, A_{1,3}, A_{2,3}]$ modulo the following relation:

$$A_{1,2} + A_{1,3} + A_{2,3} = 0,$$

so, is a free Lie algebra on two generators.

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